12.4

And if you want to divide in this way (i.e. with fractions), one <number> above another, write (lit. place) each <number> in the following way. After <you have done> this, divide one (i.e. the top one) by the other. And what remains will be a whole number.

For example: if you want to divide 20 and two parts of 13 (i.e. $20\ \frac{2}{13}$) by 3 and one third (i.e. $3\ \frac{1}{3}$), you would write the following:

$$\begin{array}{cc} 3 & 20 \\ 1 & 2 \\ 3 & 13 \end{array}$$

And now we see that $\frac{1}{3}$ is different from $\frac{2}{13}$, since they (I assume here he means $\frac{2}{13}$) does not have one third (i.e. does not directly divide into 3). Therefore, multiply the genus/type (*I think here al-Khwarizmi means denominator*) of thirds (which is 3) with the other part, (which is 13), until they are of the same genus/type/denominator. And this genus/type/denominator will be 39.

Then multiply $20\frac{2}{13}$ with 39. And 20 multiplied with 39 is 780. And also add about that $\frac{2}{13}$ths of 39, which is 6, because one thirteenth of 39 is 3. And the result <of all this> is 786.

And when you have done this, now you have the $20\frac{2}{13}$th part of the genus/type 39. After this, also multiply $3\frac{1}{3}$ with the parts which are 39, until both are of the same genus / type / denominator. And so, 3 multiplied by 39 will be 117. To this add one third of 39, i.e. 13, and the result <of all this> is 130 parts of 39. And now they have been made equal in the same genus/type/denominator. Therefore, divide one above the other, just like <i.e.> 780 over 130, and what remains will be a whole number. In truth what will have been left over will be part of the same number through which you are dividing. And it will be the (whole) number 6 and 6 parts of 130. If the numbers, which you want to divide, one through the other, are equal parts, like a quarter by a quarter or an eighth by an eighth or a seventeenth by a seventeenth, put them all back into the genus/type of that faction because they are equal. After this divide that, which you want to divide about the other, and the result of this ought to be the number one. Do this in the same way in all situations, where you want to divide fractions or whole numbers and you will find <the answer> if God is willing.

Jonathan J. Crabtree comments follow. Once again, al-Khwarizmi's example is…

$$20\frac{2}{13} \div 3\frac{1}{3}$$

Convert the quantities (improper fractions) into the same denominators (part sizes). i.e. 39ths

DIVIDEND 20 = 260/13 which is 780/39 and 2/13 = 6/39. So, altogether the dividend is 786/39

DIVISOR 3 = 9/3 which is 117/39 and 1/3 = 13/39. So, altogether the divisor is 130/39

$$20\frac{2}{13} \div 3\frac{1}{3} = \frac{786}{39} \div \frac{130}{39}$$

NOTE: Al-Khwarizmi says that because the genus/type/denominator are now the same (i.e. 39ths) the problem then becomes 786 ÷ 130, which in turn equals $6\frac{6}{130}$

In classrooms, children may be taught to convert improper fraction division into multiplication.

$$20\frac{2}{13} \div 3\frac{1}{3} = \frac{262}{13} \div \frac{10}{3} = \frac{262}{13} \times \frac{3}{10} = \frac{786}{130} = 6\frac{6}{130}$$

NOTE: Al-Khwarizmi's method is purer as it keeps the division as a division. The modern approach is numerically simpler, (in this case) yet requires a conversion of division into multiplication. "Ours in not to reason why, just invert and multiply."

What do YOU think? I'd love to know.
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P.S. To read al-Khwarizmi's explanation of fraction multiplication, goto
www.jonathancrabtree.com/mathematics/1st-english-translation-al-khwarizmi-multiplied-fractions/

[1] *Thus Spake al-Khwarizmi: A Translation of the Text of Cambridge University Library Ms. Ii.vi.5*, John N. Crossley and Alan S. Henry, Historia Mathematica, 17 (1990), 103-131. NOTE: This Latin manuscript cuts off at the start of al-Khwārizmī's explanation of fraction multiplication.

[2] To help with this project, Dr. Menso Folkerts kindly mailed and emailed me, the complete paper and digital editions of the following. *Die älteste lateinische Schrift über das indische Rechnen nach al-Ḫwārizmī*, (Latin & German) Menso Folkerts; Paul Kunitzsch; Hispanic Society of America, München : Verlag der Bayerischen Akademie der Wissenschaften, 1997.

[3] English translation above courtesy of Tess Anderson, Melbourne Australia.