The Lost Logic Of Elementary Mathematics And The Haberdasher Who Kidnapped Kaizen

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Euclid's multiplication definition from Elements, *(c. 300 BCE), continues to shape mathematics education today. Yet, upon translation into English in 1570 a 'bug' was created that slowly evolved into a 'virus'. Input two numbers into Euclid's step-by-step definition and it outputs an error. Our multiplication definition, thought to be Euclid's, is in fact that of London haberdasher, Henry Billingsley who in effect kidnapped kaizen, the process of continuous improvement. With our centuries-old multiplication definition revealed to be false, further curricular and pedagogical research will be required. In accordance with the Scientific Method, the Elements of western mathematics education must now be rebuilt upon firmer foundations.*

Multiplication Defined

Euclid's definition of multiplication first appeared in 1570 as: *A number is said to multiply a number, when the number multiplied, is so oftentimes added to itself, as there are in the number multiplying units and another number is produced,* (Billingsley, 1570). Today, the *Collins Dictionary of Mathematics* states: *to multiply **a** by integral **b** is to add **a** to itself **b** times,* (Borowski & Borwein, 2012).

We read, (Harel & Confrey, 1994):

> *In book VII, Euclid defines multiplication as 'when that which is multiplied is added to itself as many times as there are units in the other'…* and, *Mathematically, this* [i.e. multiplication] *can be represented as repeated addition (a definition found in Euclid, for example),* and, *...multiplication, as defined by Euclid, is repeated addition.*

Similar 'number added to itself multiplier times' multiplication definitions are attributed to the 11th century mathematician, Abraham bar Hiyya, and the 13th century mathematician, Jordanus de Nemore. Repeated often enough we read, (Hoffmann, Lenhard & Seeger, 2005):

> *Multiplication seems to be a simple and conceptually and epistemologically unproblematic, innocent notion. This widespread assumption is reinforced when one consults the chapter on multiplication in the famous Tropfke for algebra,... The main content of the short conceptual paragraph* [on multiplication] *is given by a reference to Euclid's definition of multiplication as repeated addition, in Book VII, definition 15; A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced (Heath 1956 II, 278; Tropfke 1980, 207-208). Multiplication thus seems to present the case of a stable notion with a meaning remaining identical over millennia.*

If he were alive today, Euclid would be *astounded*. How could people believe the above definition was his? If Euclid could speak to us he would want everyone to know the above definition of multiplication is not his idea. Instead, it is the illogical invention of Henry Billingsley, a London haberdasher.

'T is strange, - but true; for truth is always strange;
Stranger than fiction: if it could be told.
(George Gordon Byron, 1844)

Paradoxical Products

Consider the question, 'How, for example, does one add 5/8 to itself 3/4 times, $\boldsymbol{d}$ to itself π times, or –2 to itself –3 times?' (Davis, 2012). The answer is, you don't. In $\boldsymbol{ab}$, if $\boldsymbol{b} = 1$, then $\boldsymbol{a} \times 1$ or $\boldsymbol{a}$, cannot be $\boldsymbol{a}$ added to itself one time, because $(\boldsymbol{a} + \boldsymbol{a})$, equals $2\boldsymbol{a}$, not $\boldsymbol{a}$. In $\boldsymbol{ab}$, if $\boldsymbol{b} = 0$ then $\boldsymbol{a} \times 0$ or 0, is not $\boldsymbol{a}$ added to itself zero times, because $\boldsymbol{a}$ added to itself zero times is $\boldsymbol{a}$, not 0. Yet, we read, (Strogatz, 2012):

> *These humbling sessions have prompted me to revisit multiplication from scratch. And it's actually quite subtle, once you start to think about it. Take the terminology. Does "seven times three" mean "seven added to itself three times"? Or "three added to itself seven times"?*

We also read, 'The product 3 × 5 could be defined equally well as 3 + 3 + 3 + 3 + 3, i.e., 3 added to itself five times, but we have chosen to use the other convention instead: 5 added to itself three times', (Wu, 2011). So, paraphrasing Dr. Strogatz, does 'two times one' mean '2 added to itself 1 time' or '1 added to itself 2 times'? We know two times one (2 × 1) equals two, yet *see* two added to itself one time equals four (2 + 2) and *see* one added to itself two times equals three (1 + 1 + 1). Similarly, three added to itself five times equals eighteen and five added to itself three times equals twenty. Commendably, Dr. Wu has since corrected his explanations of multiplication, (HREF1). Algorithms are a 'precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps', (National Research Council, 2005). Astonishingly, Billingsley's algorithmic definition of multiplication cited since 1570 is false, because it neither commutes nor computes.

Subtracting 'Added to Itself' from Multiplication

'Added to itself' did not appear in Euclid's multiplication definition. Therefore add, added, addition and the like do not appear in any English translation of Euclid's Book VII proposition reliant upon a definition of multiplication, (HREF2). Euclid's multiplication definition had been translated into Italian, German and French by experts without a problem, yet Billingsley chose to translate the Greek συντεθῇ not as 'put/placed together', but 'added to it selfe', (HREF3). Notably, the Greek philosopher Sextus Empiricus, (c. 200 CE), had written, *To what, I ask, is at added? It cannot be added to itself, since what is added is different from that to which it is added and nothing is different from itself*, (Bury, 1933).

In 1809 William Saint suggested a correction to Euclid's definition of multiplication and in 1990 John Searle explained how to multiply correctly, yet both men were mocked, (HREF4). Instead of translating truthfully, for centuries people have been robotically inserting Billingsley's definition into mathematics texts. Just as Euclid never wrote 'added to itself' in his multiplication definition, bar Hiyya, Jordanus and Tropfke never wrote 'added to itself'. Similarly, writings of Isaac Barrow and Christian Wolff were correct yet 'scholars' inserted Billingsley's 'added to itself' phrase into their translations, (HREF5).

From MIRA to IMRA

In 2007 Keith Devlin, (HREF6), began a MIRA (Multiplication Is Repeated Addition) debate. Dr. Devlin wanted *Teachers* to stop saying 'MIRA'. Yet *Teachers* said multiplication and addition connect via the distributive law. The fact *Mathematicians* point to (with parents, employers and politicians) is mathematics rankings don't lie. English speaking countries are falling behind Asian countries, (HREF7). After the debate subsided I asked *Is Multiplication Repeated Addition?* (Table 1).

Table 1: *Is Multiplication Repeated Addition? (IMRA)*

Yes	16 votes	76.2%
No	5 votes	23.8%

2013 *Math, Math Education, Math Culture*, LinkedIn Group poll.

Several MIRA meme multiplication algorithms were then analysed. (Table 2.)

Table 2: *Multiplication Involving Repeated Addition: How Do You Calculate the Product ab?*

ab = *a* added to itself *b* times	52 votes	52.0%
ab = *a* added to zero *b* times	26 votes	26.0%
ab = *a* added to itself *b* – 1 times	22 votes	22.0%

2013 *Math, Math Education, Math Culture, The Math Connection* and *Common Core State Standards – Mathematics*, LinkedIn Group polls

Despite being historical and mathematical nonsense, Billingsley's algorithm dominates. The choice ***ab*** = ***a*** added to zero ***b*** times should have emerged by the 17th century, yet Euclid had no concept of the number *one* let alone *zero*. Infinity and zero (cypher) as numbers were problematic for European churches. Only God was infinite and the void was the realm of the devil. So to avoid cipher (zero) the 'work-around' ***ab*** = ***a*** added to itself ***b*** – 1 times, emerged.

Chinese, French and American Corrections

Soon after the emergence of the 'Billingsley Virus', (BV1570), the 17th century Chinese *Suan Fa Yuan Ben* (Elements of Calculation) was published. Euclid began his Book VII definitions with, 'An unit is that by virtue of which each of the things that exist is called one' and 'A number is a multitude composed of units', (Heath, 1908). The Chinese version read, 'One is the root of numbers' and 'A multiplicity of ones combined together is called a number'. Contrary to the writings of countless westerners, the Chinese text says three added to itself two times is nine and '...adding two to itself three times must be equal to eight', (Jami, 2012).

Seductively simplistic, BV1570 spread as if a Trojan virus hidden under the cloak of Euclid. Thus the wrong idea of multiplication spread throughout the colonies. England, an early adopter of printing presses, built a thriving export industry in the form of (infected) mathematics textbooks. Meanwhile the French disinfected the English multiplication definition. On the example of 16 × 4 we read 16 is to be repeated four times or added to itself three times, (Lacroix, 1804). After the American War of Independence the flood of infected textbooks from England slowed to a trickle. Arithmetical instantiations of multiplication increasingly took the form ***a*** × ***b*** was ***a*** taken/repeated ***b*** times, or ***a*** added to itself ***b*** – 1 times, (HREF8).

The Multiplication of Lost Logic

Mathematics promotes logical argument and deductive reasoning like, *Socrates is a man, all men are mortal, therefore Socrates is mortal.* Euclid was the master of reasoning and proof. Unlike most fields of science, Euclid's definitions and theorems are as true today as they were 2300 years ago.

Yet the *1st Law of Marketing* caused a problem and children are given worse arithmetical foundations than 1400 years ago in India, when Brahmagupta documented the rules of arithmetic that featured zero, negative numbers and 'Laws of Sign'. While Europe struggled with negative numbers in the 18th century, China had been using negative numbers for 2000 years.

Our multiplication definition has three added to itself twice being equal to six. Yet adding a number to another the same as itself is doubling and three added to itself once equals six. **Three added to itself twice equals three added to itself once!** When fractional multipliers appear, children led to believe multiplication makes more, must 'unbelieve'. How is 2 × ½ explained? When two is to be added to itself 'half a time' we arrive at 2 + 1, not ½. Similarly, children are led to believe 'division makes less' which also needs undoing when fractional divisors less than one get introduced.

Just as 'Multiplication Is Repeated Addition' (MIRA), it should be said 'Division Is Repeated Subtraction' (DIRS). Multiplication distributes over repeated addition with $2 \times {}^{+}3$ equaling $2 \times (0 + 1 + 1 + 1)$. Yet multiplication also distributes over repeated subtraction because $2 \times {}^{-}3 = 2 \times (0 - 1 - 1 - 1)$. So multiplication is also repeated subtraction. When $\boldsymbol{a} = \boldsymbol{b}$ and $\boldsymbol{b} = \boldsymbol{c}$, then logically $\boldsymbol{a} = \boldsymbol{c}$, so we must either accept the statement 'repeated addition is division' is true, or accept Billingsley's fabricated multiplication concept is false.

How Laws of Marketing Defeated Laws of Logic

The first two laws of marketing, (Ries & Trout, 1993), are said to be:

1. It's better to be first, than it is to be better, and
2. If you can't be first in a category, set up a new category you can be first in.

Billingsley was first to translate Euclid's multiplication definition into English. Yet he was fourth to translate Euclid's multiplication definition into a 'modern' language after Italian, German and French. It is possible he converted Euclid's geometrical and proportional multiplication definition into an arithmetical definition to differentiate his product. Billingsley's definition wasn't the first, yet it was the first printed and imprinted in English minds. In 1570 Roman Numerals dominated and Hindu Arabic mathematics had a steep learning curve. So rather than reveal multiplication to be Proportional Covariation, Billingsley rebranded Euclidean multiplication as repeated addition.

Billingsley's Binary Bug

When numbers were placed together as many times as a multiplier had units in the 16th century, they were repeated not across the page, but down the page. This aligned each digit for summation according to the new concept of base ten positional notation. While it is obvious now that there are two additions in 4 + 4 + 4, such a summation would have been done as shown here. Without the sign +, we can understand how 'added to itself' slipped through. People couldn't *see* the error! In four multiplied by three, written 4 + 4 + 4, four is added to four (itself) two times, not three times as said for centuries. 'Placing' is unary involving one number a time. Yet 'adding' is binary involving two numbers a time. Billingsley's 'Binary Bug' has the number of *times* the addition ***O***peration is done being the same as the number of ***T***erms, which is impossible. In binary arithmetical expressions ***n*** ***O***perations require ***n*** + 1 ***T***erms.

$$\begin{array}{r}4\\4\\\underline{4}\\12\end{array}$$

So should we update Euclid's fourth century BCE multiplication definition with India's seventh century arithmetic with one, zero and negative integers all accepted as numbers? The Scientific Method demands a YES response. Therefore, the calculation of $\boldsymbol{a} \times {}^{+}\boldsymbol{b}$ equals ***a*** added to zero (not itself) ***b*** times in succession. For integral multiplication, $\boldsymbol{a} \times {}^{\pm}\boldsymbol{b}$, we can, according to the sign of ***b***, either add ***a*** to zero ***b*** times in succession or subtract ***a*** from zero ***b*** times in succession. Such lost logic is found in, for example, the *Encyclopædia Britannica*, (Bell & Macfarquhar, 1768-1771). Under the entry for Algebra, we read:

> *Multiplication by a positive Number implies a repeated Addition: But Multiplication by a Negative implies a repeated Subtraction. And when +**a** is to be multiplied by –**n**, the Meaning is that +**a** is to be subtracted as often as there are Units in **n**: Therefore the Product is negative, being –**na**.*

Euclid's Multiplication Concept

So what did Euclid do? The answer is simple. Euclid preserved proportional relationships between four terms. As the *Unit* is to the *Multiplier*, the *Multiplicand* is to the *Product*. When *Multiplicand* ***a*** is to be multiplied by *Multiplier* ***b***, how do we arrive at *Product* ***c***? From Euclid's original multiplication definition, ***a*** is placed as many times as there are *Units* in ***b*** and the number ***c*** is *Produced*. Euclid's Book VII propositions involving multiplication saw proofs emerge via the creation of proportional line segments. Billingsley's definition of multiplication on the positive integers, corrected and clarified, is:

A number [the Multiplier] *is said to multiply a number* [the Multiplicand] *when that which is multiplied* [the Multiplicand] *is* ***placed together*** *as many times as there are units* [placed] *in the other,* [the Multiplier] *and thus some number is produced* [the Product].

Henry Billingsley was understandably a little sloppy in his translation and conversion of Euclid's multiplication definition from geometry to arithmetic. So we will look at how Euclid (implicitly) multiplied geometrically without symbolic numbers. From a line depicting the Real numbers, line segments are proportional in length to the Real Number they represent. Therefore, Euclid's definition of multiplication also applies (anachronistically) to symbolic numbers.

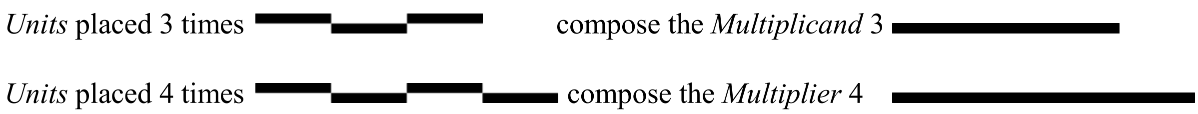
A Geometric Foundation for 3 × 4

Prior to Billingsley the application of Euclid's proportional multiplication definition had been extended from positive integers to fractions. At the start of the 16th century we find multiplication defined as *the creation of a number* [product], *being in proportion to a multiplicand, as the multiplier is to the unity*, (Huswirt, 1501). Even as the distinct nature of the multiplicand and multiplier began to blur, we still find four terms in a multiplication definition: *Multiplication is performed by two Numbers* [multiplicand and multiplier] *of like Kind for the Production of a Third,* [the product] *which will have such Reason* [ratio] *to the one, as the other hath to the Unit,* (Cocker, 1677). Isaac Newton also revealed how the proportional essence of multiplication extended well beyond the Naturals, *Multiplication is also made use of in Fractions and Surds,* [irrational roots] *to find a new Quantity in the same Ratio (whatever it be) to the Multiplicand, as the Multiplier has to Unity,* (Newton, 1720).

We read 'Multiplication is often "defined" as repeated addition, but this, I hold, is a confusion between a definition and an application', (Steiner, 2005). More specifically, if an explanation of multiplication fails to mention the four terms, *Unit/One*, *Multiplier*, *Multiplicand* and *Product*, it is an *application* of multiplication and not a *definitive Euclidean explanation* of multiplication.

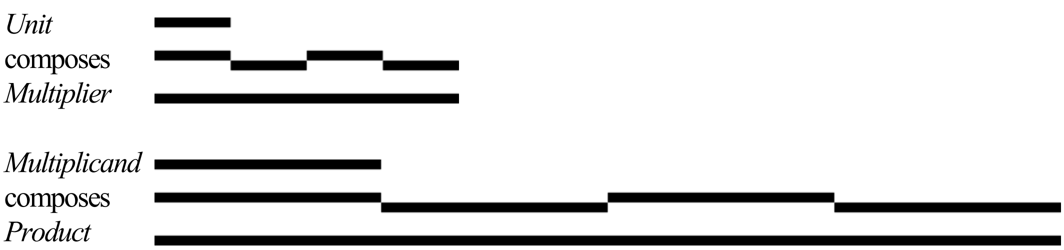
To multiply a *Multiplicand* (3) by a *Multiplier* (4), we follow Euclid.

Multiplicands and *Multipliers* are numbers composed from a given *Unit* of length. ▬ Where the *Unit* ***u*** = 1, draw ***a*** × ***b*** = ***c*** where the *Multiplicand* ***a*** = 3, the *Multiplier* ***b*** = 4 and the unknown *Product* = ***c***.



The ***Multiplicand*** *is* ***placed together*** *as many times as there are* ***Units*** *in the* ***Multiplier*** *to create the* ***Product****.*

Whatever is done with a *Unit* (1) to make the *Multiplier* (***b***), we do with a *Multiplicand* (***a***) to make the *Product* (***c***). Just as the *Unit* was placed together four times to make the *Multiplier*, the *Multiplicand* is placed together four times to make the *Product*. Via Proportional Covariation, (PCV), the three given inputs (Unit, Multiplier and Multiplicand) output the fourth term, the *Product*.



Euclid's concept is: ***As the Unit is to the Multiplier, so is the Multiplicand to the Product***, written, *Unit* **:** *Multiplier* **::** *Multiplicand* **:** *Product*. Algebraically, the four terms of the proportion are written

1 : *b* :: *a* : *c* and read, *“As* 1 *is to* ***b****, so is* ***a*** *to* ***c****”*. Multiplication, (as Proportional Covariation), means whatever we do to vary the *Unit* to make the *Multiplier,* we do to the *Multiplicand* to make the *Product*. So, with lines proportional in length to numbers, from a geometric instantiation of three (***a***) multiplied by four (***b***) we also arithmetically solve for the missing fourth term of the proportion, (***c***).

Q. What did we do to the *Unit* 1 to make the *Multiplier* ***b***?
A. We placed the *Unit* 1 together four times 1, 1, 1, 1.

Q. So what do we do to the *Multiplicand* 3 to make the *Product* ***c***?
A. We place the *Multiplicand* 3 together four times 3, 3, 3, 3.

So as 1 is to 1, 1, 1, 1, so is 3 to 3, 3, 3, 3. Put simply, 1 is to 4 as 3 is to 12 and the proportion is written 1 **:** 4 **::** 3 **:** 12. Grade 2-3 children are not ready for the proportion theory of multiplication, yet they are ready to play a proportional game of multiplication. From abstract lines we can switch to physical manipulatives. Just as a *Unit block* placed together four times in a row makes a *Multiplier*, a *Multiplicand* three *Units* long, placed together four times makes a *Product*.

Why 3 × 4 = 4 × 3

So what happens if we reverse the number of *Units* in our *Multiplicand* and *Multiplier* and draw four multiplied by three instead of three multiplied by four? Will the answer be the same? With four (***a***) multiplied by three (***b***) we again solve for the missing fourth term of the proportion, (***c***).

Q. What did we do to the *Unit* 1 to make the *Multiplier* ***b***?
A. We placed the *Unit* 1 together three times 1, 1, 1.

Q. So what do we do to the *Multiplicand* 4 to make the *Product* ***c***?
A. We place the *Multiplicand* 4 together three times 4, 4, 4.

As 1 is to 1, 1, 1, so is 4 to 4, 4, 4. Put simply, 1 is to 3 as 4 is to 12 and the proportion is 1 **:** 3 **::** 4 **:** 12. Therefore, it does not matter if, with the same *Unit*, we reverse our two conceptually different factors and write 3 × 4 or 4 × 3. That is why Euclid proved, as Proposition 13 in Book VII of Elements, *If four numbers are proportional, they will also be proportional alternately*, (Heath, 1908). Our proportion, (equality of ratios), that was 1 **:** 4 **::** 3 **:** 12 became 1 **:** 3 **::** 4 **:** 12.

Mnemonics such as *‘Minus times minus results in a plus, the reason for this, we need not discuss’* are often contrived and bad pedagogy. Yet when ***a*** × ***b*** = ***c*** is understood via the mnemonic, *Multiplying can be fun you’ll* **c***, Do to* ***a*** *as 1 made* **b**, (Crabtree, 2015), the ‘multiplication makes more’ MIRAge vanishes. With 12 × ½ we took half the *Unit* (1) to make the *Multiplier* ½ so we take half the *Multiplicand* 12 to make the *Product* 6. As 1 is to ½ so 12 is to 6 and the proportion is 1 **:** ½ **::** 12 **:** 6. With $^{-}8$ × $^{-}½$ we took half the *Unit* (1) which is ½ and changed its sign to make the *Multiplier* $^{-}½$. So we take half the *Multiplicand* ($^{-}8$) which is $^{-}4$ and change its sign to arrive at the *Product* 4. Proportionally as 1 is to $^{-}½$ so $^{-}8$ is to 4, written 1 **:** $^{-}½$ **::** $^{-}8$ **:** 4. The products of the outer terms equal the products of the inner terms.

A Chinese Connection $^{-}3 \times {}^{-}4$ (200 BCE)

As mentioned, the Chinese were comfortable with negative numbers around 2000 years before western Europe. The pedagogy of their *rod numeral* arithmetic was simple. Red rods were positive and black rods were negative, which is the opposite of our accounting convention today. Just because it may not have been done, there is nothing to stop us blending Chinese arithmetical pedagogy with Euclidean proportion theory. All we do here, for simplicity, to reveal how negative three multiplied by negative four equals positive twelve, is depict black segments as positive and red segments as negative.

How $^{-}3 \times {}^{-}4 = {}^{+}12$ Can be Depicted Geometrically

Unit
composes
Multiplier

(Units placed four times with a change of colour/sign.)

Multiplicand
composes
Product

(Multiplicands placed four times with a change of colour/sign.)

With the *Unit* fixed as positive, as 1 is to $^{-}4$, so $^{-}3$ is to $^{+}12$ and the proportion is 1 **:** $^{-}4$ **::** $^{-}3$ **:** $^{+}12$. (Confirmation of multiplicative commutativity is again left as an exercise for the reader.)

The Aftermath of BV1570

Billingsley, knighted for services as London's Lord Mayor, never wrote again on mathematics. Yet the *Billingsley brand* of 'multiplication is repeated addition' remains number 1 throughout the English speaking world. If it wasn't for Dr. Devlin, it is unlikely Billingsley's defective definition of multiplication would have been removed from the draft Common Core State Standards (CCSS) of the USA, (HREF9). Yet just as 2 + 2 + 2 is addition, the discrete area and equal group models are in the domain of addition. Only by going back to the future to 'debug' and update Euclid, will complexity be displaced with simplicity and concepts such as negative multipliers and divisors be clarified. In four-term proportions, which also alternate, the product of the two outer terms equals the product of the two inner terms. (Check with 1 **:** 3 **::** 4 **:** 12 and 1 **:** 4 **::** 3 **:** 12.) Yet as strange as it may seem, because Euclid did not consider the *Unit* to be a number and because Billingsley changed (and broke) Euclid's multiplication definition, the foundational logic of the *Unit* and *proportion* was lost.

Of course, most of the English-speaking world copes with mathematics well enough, despite Billingsley. MIRA will likely remain along with 'equal groups' and 'array' pedagogies, yet preferably within TAOMIRA, (The *Application* Of Multiplication *Involves* Repeated Addition). Via symmetry and the mathematics of the East, the true TAO, or *way* of mathematics reveals many new games and fun lesson ideas that are as profound as they are simple. Some of these have been demonstrated. PCV is a 'missing link' from both the *Naturals* to the *Reals* and elementary mathematics to physics. Arguably, the only equal to *Elements* in scientific impact is *Principia*, (Newton, 1687), which presented laws of motion, universal gravitation and more. Just as addition evaporates as proportion is revealed in Euclidean multiplication, in *Principia*, Newton mentions addition 44 times and proportion 396 times, (HREF10).

Heresy or Prophecy?

If this article appears heretical, historically, powerful people feared heretics, not because the heretics might be *wrong*, but because the heretics might be *right*. René Thom wrote: 'There is no case in the history of mathematics where the mistake of one man has thrown the entire field on the wrong track', (Thom. 1971). Yet René was wrong. He didn't know about Billingsley and neither have curriculum developers. Just as Roman Numeral arithmetic was replaced with India's, we must not rest. Generation Z will have new problems to solve so why not gift them new insights?

Because education budgets are a perennial problem, future politicians and principals may be forced to embrace the best code in class to Save Your Self/Schools/Staff/Students Time Energy and Money, (S.Y.S.T.E.M.). Just as future generations may condemn us for inaction on climate change, past generations of mathematics educators may stand condemned for their inaction on updating and fixing foundations of elementary mathematics. So, as current or future education leaders, it is our duty to *apply the Scientific Method* and pursue kaizen, or continuous improvement, to help move the human race forward, via the mathematics we teach, towards peace, prosperity and truth, for all. Thank you.

All great truths begin as blasphemies.
(George Bernard Shaw, 1919)

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UPDATES

1. A Feedback/FAQ form for this conference paper is at: https://form.jotform.co/63306463007853
2. The *Lost Logic of Elementary Mathematics* presentation, with different actionable content and classroom ideas for elementary school teachers, can be downloaded via PowerPoint Online at: https://1drv.ms/p/s!AiiJ6XgphELidETf6CoiWWpuGec