# Representations of Negative and Positive Quantities on a 'Brahmaguptan Plane' for India's Primary Classes

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**Abstract:** *Children's fear of maths is often associated with the introduction of negative numbers. By way of example, asking adult non-mathematicians for the answer to 'negative seven minus negative four' usually results in a wrong answer. However, asking the same question to 12-year-old children in the form What does seven negatives minus four negatives equal? usually results in the right answer. Why is the difference in comprehension so dramatic? In the problematic expression <u>negative seven</u> minus <u>negative four</u> the syntactic structure is <u>adjective adjective</u> verb <u>adjective adjective</u>. With the absence of a noun, the meaning of such maths for most children is lost. Instead, children (and adults) cling to rules memorised without meaning, such as 'two minuses make a plus'. So, what can we do? The answer is simple. We return to 7th Century writings of India, where we discover the astronomer Brahmagupta documented 'adjective-noun' style laws of sign, not for abstract numbers, but for positive quantities, negative quantities and zero. With this insight, we depict simple object-oriented representations of integer arithmetic involving positive and negative quantities. Such a quantitative pedagogy is concrete in nature, yet isomorphic to 'signed numbers.' Therefore, a solid intuitive foundation of integer arithmetic can be laid. Upon this foundation more abstract structures can be built. The integer teaching model that emerges is called the 'Brahmaguptan Plane'.*

## 0. Introduction

Edmund Landau, in *Foundations of Analysis* wrote, 'Please forget everything you learned in school, because you have not learned it' [1]. Perhaps if the 7th Century Indian astronomer Brahmagupta (598–670 CE) were alive today, he might say something similar.

## 1. The non-sense of debt multiplied by debt

The reader may be familiar with the mnemonic, 'Minus times minus results in a plus. The reason for this, we need not discuss' [2]. Now, the question has to be asked, how can you multiply 'minus' which is a verb, with another 'minus' which is another verb? How is an Indian child in Class 3 meant to multiply, for example, subtract two into subtract two? Similarly, with the rule negative into negative is positive, how is a child meant to multiply the adjective negative into another adjective negative to produce another adjective called positive?

In 628 CE, the Indian astronomer Brahmagupta documented the 'laws of sign', not for numbers, but for quantities. Unfortunately, an influential author depicted Brahmagupta's laws of sign as if Brahmagupta was an accountant, not an astronomer [3]. The translation appears as:

> The product or the quotient of two fortunes is one fortune.
> The product or the quotient of two debts is one fortune.
> The product or the quotient of a debt multiplied by a fortune is a debt.
> The product or the quotient of a fortune multiplied by a debt is a debt.

These four lines may be represented as follows.

| | | |
|---|---|---|
| Fortune multiplied by fortune | = fortune | $^{+}\boldsymbol{a} \times {}^{+}\boldsymbol{b} = {}^{+}\boldsymbol{ab}$ |
| Debt multiplied by debt | = fortune | $^{-}\boldsymbol{a} \times {}^{-}\boldsymbol{b} = {}^{+}\boldsymbol{ab}$ |

| | | |
|---|---|---|
| Debt multiplied by fortune | = debt | ${}^{-}\boldsymbol{a} \times {}^{+}\boldsymbol{b} = {}^{-}\boldsymbol{ab}$ |
| Fortune multiplied by debt | = debt | ${}^{+}\boldsymbol{a} \times {}^{-}\boldsymbol{b} = {}^{-}\boldsymbol{ab}$ |

The above equations may be correct yet their accompanying explanations are not. A mortgage on a home is a debt and that debt is a noun owned by the home-owner. Similarly, people around the world often carry debt on their VISA or Mastercard. That debt is also a noun. Now, according to the dubious translation of Brahmagupta's Sanskrit, if a home-owner multiplied his mortgage debt into his credit card debt, he will no doubt be pleased with the fortune produced! Of course, such an idea is nonsense. In Integer arithmetic, you cannot multiply a verb into a verb, an adjective into an adjective, or a noun into a noun. The financial interpretation of Brahmagupta's astronomical writings in the West has been around for more than a century. We read [4] 'The Indians … brought out the difference between positive and negative quantities by attaching to the one the idea of 'possession,' to the other that of 'debts'.

The Indian born mathematician and founder of the London Mathematical Society Augustus De Morgan, addressed the issue of multiplying nouns into nouns with the following statement: '… the vice of confounding abstract and concrete number, which leads him to imply that five shillings can be multiplied by [into] five shillings, runs through his whole book...' [5]

## 2. Grammar and Brahmagupta's laws of sign

Brahmagupta's sign laws for multiplication involved both concepts of the 'dash symbol –' used today for the adjective 'negative' and the verb 'subtract'. For clarity, we will use a superscript dash as a negative sign and a standard dash as a sign of subtraction. In a section titled 'calculations dealing with quantities bearing positive and negative signs and zero [6] Brahmagupta defined zero as the sum of equally positive and negative quantities. Today, we might write this as $0 = {}^{+}\boldsymbol{n} + {}^{-}\boldsymbol{n}$ and thus arrive at the equation $0 - {}^{-}\boldsymbol{n} = 0 + {}^{+}\boldsymbol{n}$.

A negative quantity subtracted from zero creates the same result as a positive quantity added to zero. Multiplication distributes over subtraction as well as addition, so we have for example, ${}^{-}2 \times -3$ being equal to $0 - {}^{-}2 - {}^{-}2 - {}^{-}2$. Given zero as defined by Brahmagupta equals $({}^{+}2 + {}^{+}2 + {}^{+}2) + ({}^{-}2 + {}^{-}2 + {}^{-}2)$, we can rewrite $0 - {}^{-}2 - {}^{-}2 - {}^{-}2$ as $({}^{+}2 + {}^{+}2 + {}^{+}2) + ({}^{-}2 + {}^{-}2 + {}^{-}2) - ({}^{-}2 + {}^{-}2 + {}^{-}2)$ which equals $({}^{+}2 + {}^{+}2 + {}^{+}2)$ or ${}^{+}6$. By way of example, a debt of 20,000 rupees taken away from you three times has the same effect as you being given 20,000 rupees three times. Thus, the law of sign for multiplication is neither minus into minus is plus, nor negative into negative is positive. The grammatically correct concept is 'a negative quantity subtracted is a positive quantity'. It is possible the confused use of financial terms such as debt and fortune arose because the Sanskrit ḍhana can mean wealth, positive or plus. Similarly, the Sanskrit ṛṇa (RiNa) can mean debt, negative or minus [7].

With the + symbol also having a dual role as meaning either the adjective positive or the verb add, we have four kinds of operations (verbs) acting on quantities (nouns) commencing from zero. We can add positive quantities to zero, add negative quantities to zero, subtract positive quantities from zero and subtract negative quantities from zero. With integer multiplication the four cases are: 1) Positives repeatedly added to zero, 2) Positives repeatedly subtracted from zero, 3) Negatives repeatedly added to zero, and 4) Negatives repeatedly subtracted from zero.

$${}^{+}\boldsymbol{a} \times +\boldsymbol{n} = {}^{+}\boldsymbol{na} \qquad {}^{+}\boldsymbol{a} \times -\boldsymbol{n} = {}^{-}\boldsymbol{na} \qquad {}^{-}\boldsymbol{a} \times +\boldsymbol{n} = {}^{-}\boldsymbol{na} \qquad {}^{-}\boldsymbol{a} \times -\boldsymbol{n} = {}^{+}\boldsymbol{na}$$

In the above, the multiplicand ***a*** represents a count or measure of either a negative quantity or a positive quantity. The multiplier ***n*** represents a count of the number of times the given quantity is either added to zero or subtracted from zero. We can then map the multiplicands representing opposing quantities to a horizontal axis that is symmetric about 0. Then, we can map the multipliers representing counts of addition or subtraction to a vertical axis, also symmetric about 0. The result is the **Brahmaguptan Plane** upon which we can depict positive products in Quadrants I and III and negative products in II and IV as depicted below.

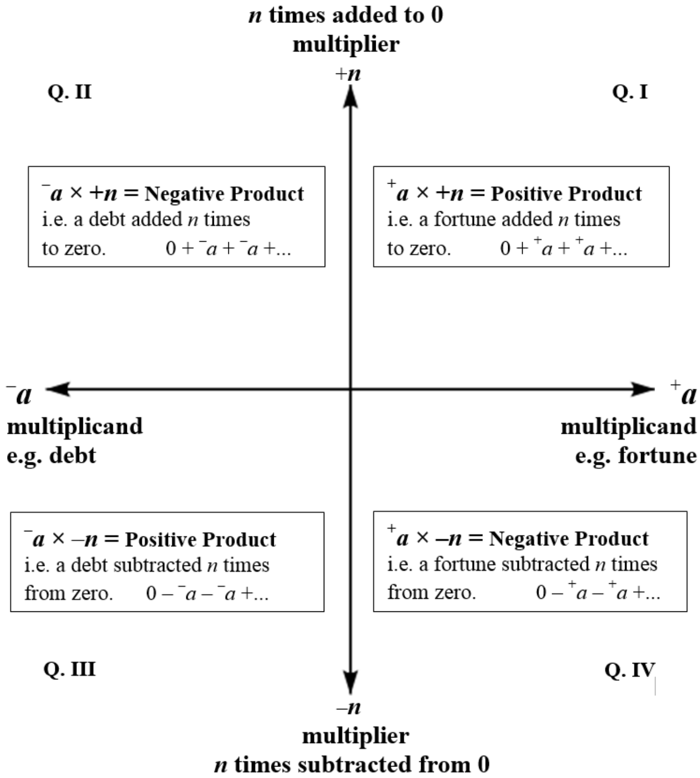


**Figure 2.1** The Brahmaguptan Plane with explanations of the laws of sign for multiplication

Zero can be defined via Brahmagupta in the form 0 = 1Neg + 1Pos. Now, we no longer see $-1 \times -1$ as 'minus one into minus one' or 'negative one into negative one'. How can we explain $-1 \times -1 = +1$ when we can't even vocalise the equation correctly with the correct words, let alone understand it? Instead, ***an*** is now defined as ***a*** quantities either added to zero ***n*** times or ***a*** subtracted from zero ***n*** times, according to the sign of ***n***. Therefore, $-1 \times -1$ is rewritten $^{-}1 \times -1$ so it can be read as 'negative one into minus one', which means one negative subtracted from zero one time. Given 0 = 1Neg + 1Pos, subtracting one negative from zero one time simply leaves one positive, thus demonstrating why $^{-}1 \times -1 = {}^{+}1$.

## 3. Negative area models

We read, 'Negative numbers troubled mathematicians far more than irrational numbers did, perhaps because negatives had no readily available geometrical meaning and the rules of operation were stranger' [8] Now that we have associated quantities as nouns being either added to or subtracted from zero to produce other noun quantities, either of the same kind or opposite kind, we have moved away from the western myth that it is possible to have quantities that are 'less than zero'. Mathematics is primarily about relationships between quantities and numbers simply represent counts or measures of quantity. In physics, it is impossible to have any quantity less than zero. We do, however, have laws such as for every action there is an equal and opposite reaction. The number of electrons with a negative charge is never less than zero, as the number of positive positrons is never less than zero.

A bias has existed in geometry for millennia. Geometry is derived from *geo* meaning earth and *metry* meaning measure. Earth measurement was necessary for ancient Egypt whenever the river Nile flooded. If you had less land to grow your crops because of flooding, you would be taxed less. The ancient Greek Thales of Miletus learnt some of his geometry skills from the Egyptians and brought them back to Greece where the practical art of earth-measurement evolved into abstract geometry.

Yet, what might have happened had the Egyptians been fishermen? A reduced area of water would have meant there was less opportunity to cast fishing nets. Instead of geo-metry the Greeks might have written about hydro-metry or water measurement. Along the Nile, as the area of water increased the area of land decreased and vice-versa. Thus, we have a natural dual quantitative system in which the farmers with their crops might consider areas of land positive and areas of water negative, whilst the fishermen with their nets might consider the area of water positive and areas of land to be negative.

In the 17$^{th}$ Century, it was an arbitrary choice the Englishman John Wallis made to choose a rightward direction to represent positive movement and a leftward direction to represent negative movement [9]. Had the English been at war with Ireland instead of the Dutch, it's entirely possible his analogy could have depicted positive directions to the left and negative directions to the right. Thus, it will not matter which of two opposing quantities is deemed to be positive, relegating the other negative quantity to the role of additive inverse, where equal numbers of opposing units sum to zero. The only thing that matters is that once a choice has been made regarding which of the two opposing quantities is named positive, the convention be adhered to. From the Brahmaguptan Plane introduced previously, we can now depict both positive areas and negative areas on the plane as depicted below.

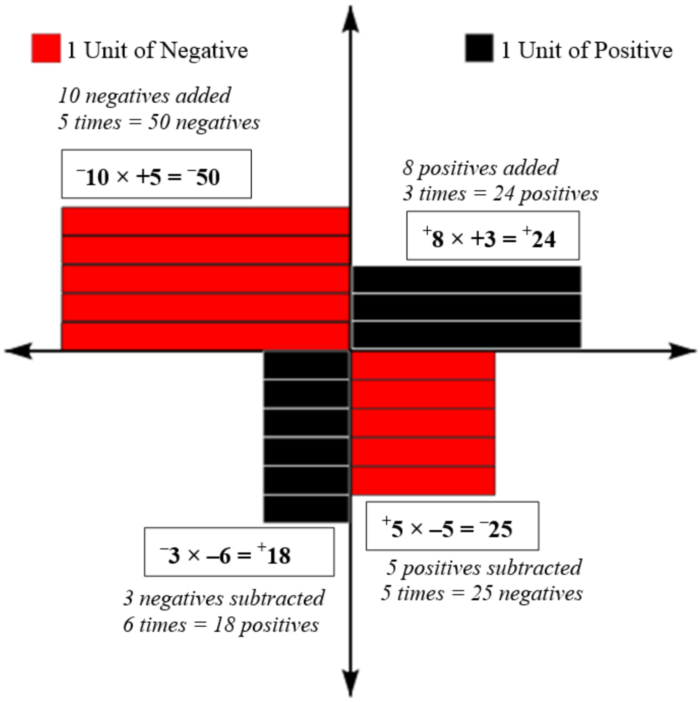

**Figure 3.1** The Brahmaguptan Plane with both positive and negative areas

## 4. Matching grammar with physical models

Colour coding diagrams or even classroom manipulatives to identify positive and negative quantities returns missing noun concepts to the physical foundations of mathematics. It should be noted that in ancient Greek mathematics, neither zero nor one were considered numbers and the idea of negative quantities had not taken hold.

Perhaps too often, primary level mathematics education follows an abstract path, where laws and rules are memorised without meaning. Teachers may say 'We define negative into negative equals positive in order that the distributive property holds' without realising the reason negative subtracted is positive stems from Brahmagupta's breakthrough definition of zero as the sum of two equal yet opposite quantities. Memorising 'negative into negative is positive' alone will not develop deep insights that will be of practical use to solve simple day-to-day problems or more complex engineering problems. Āryabhaṭa (476–550 CE) who spread the concept of base ten place notation within India and Brahmagupta who documented the laws of sign for positive quantities, negative quantities and zero were after all, empirical astronomers, not abstract philosophers.

Primary level teachers often do not understand what is written on a page. A common lesson involves simplifying terms in an expression. For example, with the expression 24 – 3 + 2 – 5 + 7 – 1 + 6, the usual advice is to group all the positive terms together and then group all the negative terms together. Thus, the student arrives at 24 + 15 – 9 = 30. However, there are no negative terms in this expression, simply positive terms being added and positive terms being subtracted. Teachers often use an expression such as (10 – 2) × (10 – 3) to 'prove' that negative two into negative three equals positive six, because (***a*** – ***b***) × (***c*** – ***d***) = ***ac*** – ***ad*** – ***bc*** + ***bd***. Yet once again the numbers are being misread. There are no negative numbers in the above expression. In *Arithmetica*, (c. 250 CE), Diophantus wrote (in Greek), 'A wanting multiplied by a wanting makes a forthcoming' [10]. Looking at the geometry, we see that the fourth term ***bd*** above arises from subtracting too much and having to add some area back as shown below. In (***a*** – ***b***) × (***c*** – ***d***) = ***ac*** – ***ad*** – ***bc*** + ***bd*** from ***ac*** on the far left, we subtract the white top horizontal area ***ad*** in the middle diagram. Then the white right vertical area ***bc*** is subtracted in the rightmost diagram below. However, the shaded area ***bd*** has been subtracted twice, thus it must be added back, resulting in the final instruction + ***bd***.

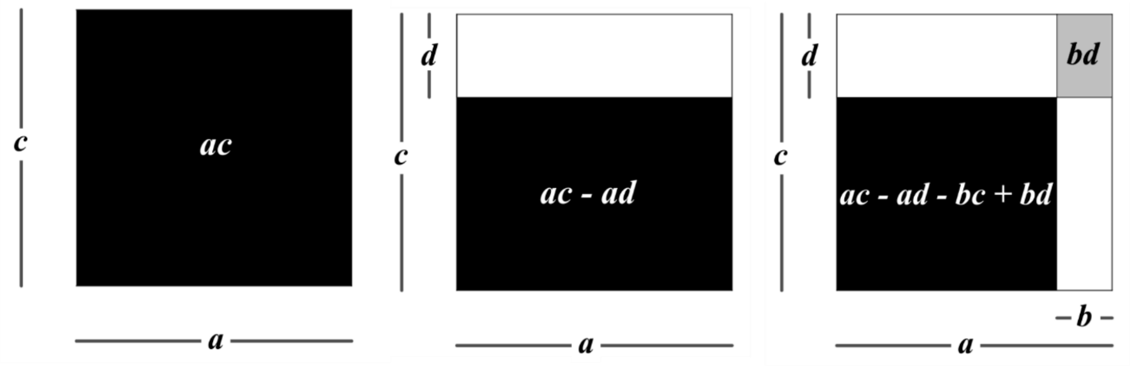


**Figure 4.1** How '+ ***bd***' is created by a subtracted quantity subtracted twice being added back

The expression (10 – 2) × (10 – 3) is just a convoluted way of writing 8 × 7. To multiply two 'dashed numbers', we must present a geometrical solution for (2 – 10) × (3 – 10). First, we convert the expression into a multiplicand comprising positive and negative quantities, so (2 – 10) becomes ($^{+}2$ + $^{-}10$). Then the adverbial multiplier which counts how many times quantities are added or subtracted from zero appears as [(+3) + (–10)]. The steps for ($^{+}2$ + $^{-}10$) × [(+3) + (–10)] are in the Table below.

**Table 4.1** The step-by-step workings for ($^{+}2$ + $^{-}10$) × [(+3) + (–10)]

| | | | |
|---|---|---|---|
| **Q1**. $^{+}2$ × (0 + 3)<br>2 Pos Added<br>3 Times onto Zero<br>makes 6 Pos | **Q2**. $^{-}10$ × (0 + 3)<br>10 Negs Added<br>3 Times onto Zero<br>makes 30 Negs | **Q3**. $^{-}10$ × (0 – 10)<br>10 Negs Subtracted<br>10 Times from Zero<br>makes 100 Pos | **Q4**. $^{+}2$ × (0 – 10)<br>2 Pos Subtracted<br>10 Times from Zero<br>makes 20 Negs |

Thus, from (2 – 10) × (3 – 10), in the four quadrants noted above and pictured below, we have identified; 6 positives, 30 negatives, 100 positives and 20 negatives, which simplifies to 106 positives and 50 negatives, which together, reduce to 56 positives, or $^{+}56$, which happens to be $^{-}8$ × $^{-}7$.

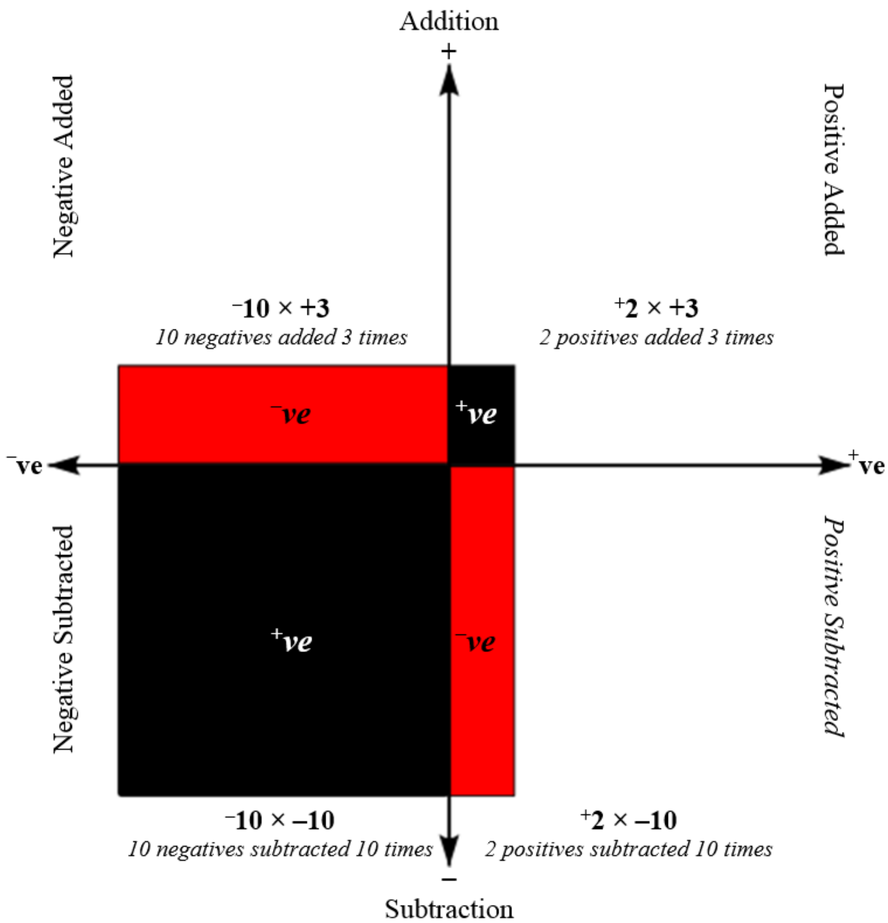


**Figure 4.2** A visual instantiation of (2 – 10) × (3 – 10) with both positive and negative areas.

## References

[1] Landau, E. (1930). *Grundlagen der Analysis*. Leipzig: Translation via Landau, E., & Steinhardt, F. (1966). *Foundations of analysis the arithmetic of whole, rational, irrational and complex numbers.* New York: Chelsea House Publishers.
[2] Carpenter, H. (1981). *W. H. Auden: A biography*. London: Allen & Unwin.
[3] Ifrah, G., & Bellos, D. (1998). *The universal history of numbers: From prehistory to the invention of the computer*. London: Harvill Press. NOTE Ifrah's book actually has, 'the product or the quotient of two debts is one debt', which is a translation error from the French.
[4] Cajori, F. (1893). *A history of mathematics*. New York: Macmillan.
[5] Morgan, A. (1847). *Arithmetical books from the invention of printing to the present time, being brief notices of a large number of works drawn up from actual inspection, by Augustus de Morgan*. London: Taylor and Walton.
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[10] Diophantus, & Heath, T. L. (1910). *Diophantus of Alexandria: A study in the history of Greek algebra*. Cambridge: Univ. Press.