A NEW MODEL OF MULTIPLICATION VIA EUCLID

This article was prompted by Andrew Wrigley's How to teach an old curve new tricks. *Vinculum* 52(2) (April 2015).

For the past 446 years, many people in the West have been misled by the 'MIRA' multiplication myth, 'Multiplication Is Repeated Addition'. It is thought the Greek mathematician, Euclid of Alexandria, defined multiplication as repeated addition around 300 BCE, yet he didn't. Words such as 'add' and 'added' do not appear in any of Euclid's Book VII propositions reliant on a definition of multiplication. The Greek word συντεθῇ in Euclid's definition was incorrectly translated in 1570 by Henry Billingsley as 'added to itself' instead of 'placed together'. Thus, illogical definitions of multiplication appear, such as in *Collins Dictionary of Mathematics*, 'To multiply a by integral b is to add a to itself b times.'

CORRECTING THE DEFINITION OF MULTIPLICATION

The first translation into English of Euclid's definition of multiplication occurred in 1570 by the London haberdasher Henry Billingsley. This incorrectly read, 'A number is sayd to multiply a number, when the number multiplyed, is so oftentimes *added to it selfe*, as there are in the number multiplying unities: and an other number is produced.' The correct translation should read, '… when the number multiplied is placed together as many times as there are …'

In 1968, when my Year 2 teacher asked, '*Who can tell me what two added to itself three times is?*' I was surprised to discover the answer wasn't eight! Yet a seventeenth Century Chinese text said, '...adding two to itself three times must be equal to eight'. If $b = 1$, then $a \times 1$ cannot be a added to itself one time, because a added to itself one time equals $2a$, not a. More significantly, on multiplication as repeated addition, mathematics professor, Dr. Brent Davis, recently asked, 'How, for example, does one add 5/8 to itself 3/4 times, d to itself π times, or –2 to itself –3 times?' You don't!

The same problem exists with the definition of powers. Newton, in Latin, said a^3 was a multiplied by itself twice. Yet *Collins Dictionary of Mathematics* is among many today who define a cube as, 'The result of multiplying a number, quantity or expression by itself three times.'

MULTIPLICATION AS PROPORTIONAL COVARIATION (PCV)

Modern mathematics is said to have began in the seventeenth Century with the ideas of René Descartes which led to analytic geometry and the Cartesian Plane. Notably, in *Discours de la methode* (1637), an appendix appeared with the title, *La Géométrie*. The first diagram in *La Géométrie* depicted the multiplication of line segments via similar triangles.

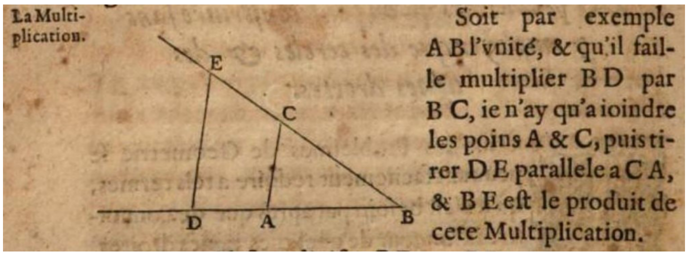

Figure 1. Image from La Géométrie, p. 298. Translation: For example, let AB be taken as unity, and let it be required to multiply BD by BC, then I have only to join the points A and C, and draw DE parallel to CA; and BE is the product of this Multiplication.

The diagram above of Descartes first appeared some 2000 years earlier, as Proposition 12 in Book VI of Euclid's *Elements*; 'To find a fourth proportional to three given straight lines'. The depiction of multiplication via similar triangles leveraged the fact ratios of corresponding side lengths are equal. So the multiplicative proportion is *Unit : Multiplier (BC) = Multiplicand (BD) : Product (BE)*. The innovation of Descartes was to admit unity as one of the three given straight lines under consideration.

Isaac Newton's *Arithmetica Universalis* (1707) presented the same geometric depiction of multiplication.

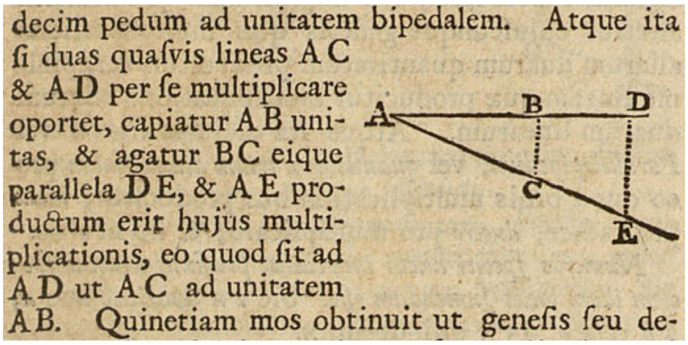

Figure 2. Image from page five of Arithmetica Universalis. Translation: If you were to multiply any two Lines, AC and AD, by one another, take AB for Unity, and draw BC, and parallel to it DE, and AE will be the product of this multiplication, because it [AE] is to AD as AC, [is] to AB Unity.

Notably, Newton also wrote,

> Multiplication is also made use of in Fractions and Surds, to find a new Quantity in the same Ratio (whatever it be) to the Multiplicand, as the Multiplier has to Unity.

and

> In Geometry, if a line drawn any certain way be reckon'd for affirmative, then a line drawn the contrary way may be taken for negative: As if AB be drawn to the right; and BC to the left; and AB be reckon'd affirmative, then BC will be negative…' *Universal Arithmetic*, pp. 3-4.

Diagrams depicting multiplication with lines drawn in a contrary, or negative way, might have emerged, yet didn't.

A CLOSER LOOK AT INTEGER MULTIPLICATION

Zero as a number emerged long after Euclid, yet is central to our number line. With integers, the MIRA mirage evaporates.

ADDITION OF A POSITIVE NUMBER

$a + +b$, when $b = 4$:

Start at a, and count up/right 1 (the unit of count) 4 times

$a + 4 = a + 1 + 1 + 1 + 1$

SUBTRACTION OF A POSITIVE NUMBER

$a - +b$, when $b = 4$:

Start at a, and count down/left 1 (the unit of count) 4 times

$a - 4 = a - 1 - 1 - 1 - 1$

MULTIPLICATION BY A POSITIVE NUMBER

$a \times +b$, when $b = +4$:

Start at 0, and count up/right a (the new unit of count) 4 times

$a \times +4 = 0 + a + a + a + a$

MULTIPLICATION BY A NEGATIVE NUMBER

$a \times -b$, when $b = -4$:

Start at 0, and count down/left a (the new unit of count) 4 times

$a \times -4 = 0 - a - a - a - a$

To make the multiplier $+\boldsymbol{b}$, we took the unit 1, $\boldsymbol{b}$ times, so we take the multiplicand **a**, $\boldsymbol{b}$ times, to get the product $+\boldsymbol{ab}$. For the multiplier $-\boldsymbol{b}$, we took away the unit 1, $\boldsymbol{b}$ times, so we take away the multiplicand **a**, $\boldsymbol{b}$ times, to get the product $-\boldsymbol{ab}$. From these proportional covariations, new algorithmic recipes emerge for integer multiplication.

$\boldsymbol{a} \times +\boldsymbol{b}$ To multiply $\boldsymbol{a}$ by $+\boldsymbol{b}$, is to add $\boldsymbol{a}$ to zero $\boldsymbol{b}$ times in succession

$\boldsymbol{a} \times -\boldsymbol{b}$ To multiply $\boldsymbol{a}$ by $-\boldsymbol{b}$, is to subtract $\boldsymbol{a}$ from zero $\boldsymbol{b}$ times in succession

REAL MULTIPLICATION VIA INTERSECTING CHORDS

Multiplicative proportions are preserved, regardless of the signs of the factors, and regardless of their nature; integral, rational or irrational.

Euclid's Intersecting Chords Theorem, overlaid atop a Cartesian Plane, presents an instantiation of proportional covariation (PCV) on the set of real numbers.

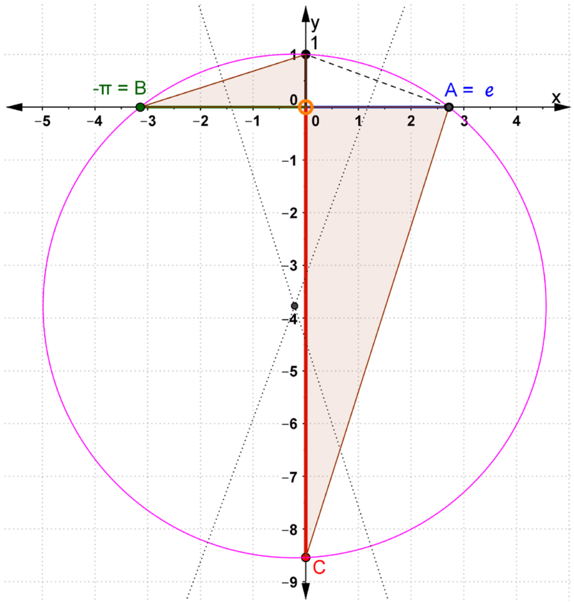

Figure 3. Demonstration of multiplication using PVC.

Figure 3 uses GeoGebra to depict $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}$ where $\boldsymbol{A}$ is e and $\boldsymbol{B}$ is $-\pi$. The point 1 (0, 1) is fixed and points $\boldsymbol{A}$ and $\boldsymbol{B}$ can be moved along both sides of the x-axis. No matter what two distinct real numbers are plotted on the x-axis, the fourth proportional product, $\boldsymbol{C}$, will always be found on the y-axis, determined by the unique circle constructed from 1, $\boldsymbol{A}$ and $\boldsymbol{B}$. The centre of the circle can be found by defining the intersection of the perpendicular bisectors of the chords $A1$ and $B1$.

A PROOF OF THE PCV DEMONSTRATION

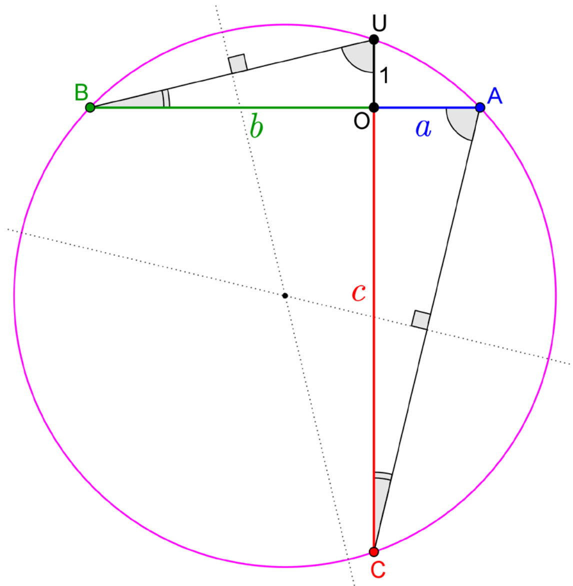

Figure 4.

$\angle BUC = \angle BAC$, both subtended by the arc BC.

$\angle UBA = \angle UCA$, both subtended by the arc UA.

$\angle UOB = \angle AOC$, as intersecting straight lines make equal opposite angles.

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ΔOBU and ΔOCA are similar, as their angles are the same.

Therefore $OA : OC = OU : OB$.

Therefore $OA \times OB = OC \times OU$.

As OU is 1, then $OA \times OB = OC$ and $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}$.

WHAT IS MULTIPLICATION REALLY?

The true meta multiplication model (PCV) has been missing for centuries. In 1501 the German mathematician, Johann Huswirt, defined multiplication as, 'the creation of a number (product), being in proportion to a multiplicand, as the multiplier is to the unity'. Division is the creation of a quotient, being in proportion to a dividend, as the unit is to the divisor. In 1677, the Englishman Edward Cocker, author of *Cocker's Arithmetick,* began his chapter on multiplication with the following.

> Multiplication is performed by two Numbers of like Kind for the Production of a Third, which will have such Reason (ratio) to the one, as the other hath to the Unit.

Cocker was one of the last popular mathematics writers to prioritise the proportional nature of multiplication, before proceeding to its applications.

CONCLUSION

In the meta multiplication model, proportional covariation, PCV, there are three inputs, not two. They are the number to be multiplied (the multiplicand a) the number doing the multiplying (the multiplier b) and the unit of count or measure (the unit, 1). These three real numbers output c, the fourth proportional product. Multiplication on the real numbers simply involves finding a product that has the same ratio to the multiplicand as the multiplier has to 1. With straight lines proportional to the numbers they represent, such products can be constructed with a circle. Repeated addition/subtraction, arrays, area models, scaling and other applications from the multiplicative conceptual field are all subsets of PCV.

APPENDIX

The axiomatic treatments of Grassman, Dedekind, Peano and Landau never defined ab. They only ever defined $a(b + 1)$ as $ab + a$ when $a \times 1 = a$. To get to a definition of ab, we write

$ab + a = ab + a$

$ab = ab + a - a$ (subtract a from both sides)

$ab = a + ab - a$ (commutative law)

$ab = a + a(b - 1)$

This reads ab equals a added to itself $(b - 1)$ times, not b times.

To the best of my knowledge, only Henri Poincaré defined ab to arrive at the above definition, which contradicted and corrected Euclid à la Billingsley.

Interestingly, Henry Billingsley decided to disregard his own definition! In his notes that followed his definition he explained multiplication as follows.

> 3 taken... 4 tymes maketh 12. For 3 foure tymes is 12
> 2… taken 4 tymes... maketh 8
> 3 taken twice maketh 6
> 3 taken… 3 tymes maketh 9 and
> 2 taken 3 times maketh 6.

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RESOURCES

Henry Billingsley's incorrect translation of Euclid's definition of multiplication. www.jonathancrabtree.com/mathematics/henry-billingsleys-definition-multiplication-euclids-elements/

Scans of correct translations of Euclid's definition of multiplication in Italian, German and French. www.jonathancrabtree.com/euclid/elements_book_VII_definitions.html

An Introduction to PCV (Proportional Covariation). Interactive applets that may be downloaded and edited, including multiplication and division, via Euclid's intersecting chords theorem. http://tube.geogebra.org/book/title/id/1410639

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